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Infinitely Many Contact Structures on S^{4m+1}

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1 Introduction and main results

A *contact manifold* (M, ξ) is a smooth $(2n-1)$-dimensional manifold M with a completely nonintegrable hyperplane distribution $\xi \subset TM$. This means that locally ξ is defined by a 1-form α satisfying $\alpha \wedge (d\alpha)^{n-1} \neq 0$. Such a distribution ξ is called a *contact structure*. When ξ is coorientable (for instance, when M is simply connected), α exists globally and is called a *contact form*. In this case, the structure group of TM reduces to $\mathbf{U}(n-1) \times 1$ in the following way. First, the Reeb *vector field* $R = R_\alpha$, which is transverse to ξ, is determined by the equations

$$i_R d\alpha = 0, \qquad i_R \alpha = 1.$$

Next, on ξ, one can choose an almost complex structure J compatible with the symplectic form $d\alpha|_\xi$ in the sense that $d\alpha(X, JY)$ defines a Riemannian metric on ξ. This almost complex structure is uniquely determined up to homotopy; in fact, it is determined by the cooriented contact structure since the conformal class of $d\alpha|_\xi$ depends only on ξ, not the particular choice of α.

Such a reduction of the structure group of TM is called an *almost contact structure*. Let $[\xi]$, the *formal homotopy class* of ξ, be the homotopy class of almost contact structures associated with ξ. We see that $[\xi]$ is a global invariant of the contact structure ξ. It is appropriate to mention here that, by Darboux's theorem, contact structures, like symplectic ones, admit no local invariants.

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The basic questions of contact geometry are those of existence and classification of contact structures. In dimension 3, these questions have been studied extensively by Martinet, Lutz, Bennequin, Eliashberg, Giroux, and others. In higher dimensions, still very little is known, especially regarding the classification problem. In this paper, we are concerned with contact structures on spheres, all of which are necessarily coorientable. Let us summarize some of the previously known results.

On S^3, cooriented contact structures were completely classified by Eliashberg in [4]. In addition to the standard (*tight*) structure, there is an *overtwisted* one in each homotopy class of tangent 2-plane distributions on S^3. These classes are distinguished by the Hopf invariant, which is an element of $\pi_3(S^2) \cong \mathbf{Z}$.

Sato [13] showed that the following hold.

(a) On S^{4m-1} ($m > 1$), there exist infinitely many contact structures representing different homotopy classes of almost contact structures.

(b) On S^{4m+1}, each homotopy class contains a contact structure. This gives, however, only finitely many structures on spheres of this dimension.

Based on Gromov's theory of J-holomorphic curves, Eliashberg [3] found an exotic contact structure in the standard homotopy class on each S^{4m+1}. Using similar arguments, Geiges [6] constructed a homotopically standard exotic contact structure on S^7. It is still unknown whether such a structure exists in dimensions $4m - 1$ for $m \geq 3$.

We say that two contact structures on the same manifold are *isomorphic* if there exists a diffeomorphism of the underlying manifold that transforms one into another. By a theorem of Gray [7], any homotopy of contact structures is generated by an isotopy. Thus the notion of an isomorphism is weaker than that of a homotopy. The purpose of this paper is to prove the following result.

Main theorem. For each natural number m, there exist infinitely many pairwise nonisomorphic contact structures on S^{4m+1} in every homotopy class of almost contact structures. $\square$

The paper is organized as follows. In the next section, we describe a certain set of contact structures on S^{4m+1}. Our construction, like Sato's, is based on Brieskorn's model for homotopy spheres and is essentially the same as in [8] or [12]. These contact structures are distinguished using a new global invariant of contact manifolds, called contact homology. A brief introduction into the theory of contact homology is given in Section 3. Finally, in Section 4, we prove the main theorem by computing the contact homology of the contact structures in question.

2 Brieskorn manifolds

The Brieskorn manifold $\Sigma(a) = \Sigma(a_0, \dots, a_n)$ is defined as the intersection of the singular hypersurface

$$z_0^{a_0} + \dots + z_n^{a_n} = 0$$

in $\mathbf{C}^{n+1}$ with the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$. Here a_j are natural numbers, $a_j \geq 2$. This is a smooth $(2n-1)$-dimensional manifold, and by [8], it admits a contact form

$$\alpha = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\overline{z_j} - \overline{z_j} dz_j).$$

The corresponding Reeb vector field

$$R_\alpha = \left(\frac{4i}{a_0} z_0, \dots, \frac{4i}{a_n} z_n \right)$$

has only periodic trajectories and hence defines an effective S^1-action on $\Sigma(a)$.

From now on, we restrict ourselves to the special case

$$a_0 = p, \;\; a_1 = \dots = a_n = 2,$$

where $n = 2m+1$ is odd and $p \equiv \pm 1 \bmod 8$. As shown in [1], the manifold $\Sigma(a) = \Sigma(p, 2, \dots, 2)$ is diffeomorphic to the standard sphere S^{4m+1}. Therefore, we obtain infinitely many contact structures ξ_p on S^{4m+1} given by contact forms

$$\alpha_p = \frac{ip}{8}(z_0 d\overline{z_0} - \overline{z}_0 dz_0) + \frac{i}{4} \sum_{j=1}^{n} (z_j d\overline{z_j} - \overline{z_j} dz_j).$$

Theorem 2.1. For $p_1 \neq p_2$, the structures ξ_{p_1}, ξ_{p_2} are not isomorphic. □

The main theorem follows from this statement. Indeed, homotopy classes of almost contact structures on S^{4m+1} are classified by $\pi_{4m+1}(\mathbf{SO}_{4m+1}/\mathbf{U}_{2m})$. This group, as computed by Massey [9], is cyclic of order

$$d = \begin{cases} (2m)!, & \text{if } m \text{ is even}, \\ (2m)!/2, & \text{if } m \text{ is odd}. \end{cases}$$

According to the computations by Morita [10], the formal homotopy class of the contact structure ξ_p is given by the formula

$$[\xi_p] = (p-1)/2 \bmod d.$$

Unless $m = 1$ (in which case the group is trivial), d is divisible by four; and so the structures ξ_p represent half of the homotopy classes given by the elements of $\mathbf{Z}_d$ congruent to 0 or 3 modulo 4. In particular, for $p \equiv 1 \bmod 2(2m)!$, the contact structures ξ_p are homotopically standard. The other half of the homotopy classes is represented by the structures $f^*\xi_p$, where f is an orientation-reversing diffeomorphism of S^{4m+1}.

Theorem 2.1 is proved in Section 4.

3 Contact homology

Contact homology is a Floer-type invariant of contact manifolds recently introduced by Y. Eliashberg and H. Hofer. In this section, we sketch its definition and main properties. For more details, the reader is referred to [5].

Let (M, ξ) be a contact manifold. For simplicity, we assume that $\pi_2(M) = 0$. Fix a contact form α with $\ker \alpha = \xi$. We are interested in studying periodic trajectories of the Reeb vector field R_α. First, recall that such a trajectory γ is called *nondegenerate* if the linearized Poincaré return map along γ has no eigenvalues equal to 1. If γ is in addition contractible, one can define its *Conley-Zehnder index* $\mu(\gamma)$ as follows. Choose a disc D spanning γ in M, and trivialize the symplectic vector bundle $(\xi, d\alpha)$ over D. The linearized Reeb flow along γ defines then a path in the linear symplectic group $\mathbf{Sp}(2n-2)$, connecting identity to a matrix with all eigenvalues different from 1. We set $\mu(\gamma)$ to be the Conley-Zehnder index of this path (see [2], [11]). Since the second homotopy group of M vanishes, this definition does not depend on the choice of the spanning disc D. For technical reasons, it is convenient to use the *reduced* Conley-Zehnder index $\tilde{\mu}(\gamma) = \mu(\gamma) + n - 3$.

Let $\mathcal{P} = \mathcal{P}_\alpha$ be the set of all contractible periodic trajectories of R_α. We do not fix initial points on the trajectories and include all multiples as separate points of $\mathcal{P}$. For a generic choice of the contact form α, each trajectory in $\mathcal{P}$ is nondegenerate and there are only countably many of them.

Now consider a free super-commutative graded algebra $\Theta = \Theta_\alpha$ over $\mathbf{C}$ with the unit element generated by elements of $\mathcal{P}$. Here the grading of the generators is given by their reduced Conley-Zehnder index; i.e., $\deg(\gamma) = \tilde{\mu}(\gamma)$, and the super-commutativity means

$$ab = (-1)^{\deg(a)\deg(b)} ba.$$

In other words, Θ is a polynomial algebra on generators of even degree, and an exterior algebra on generators of odd degree.

In addition, fix an almost complex structure J on ξ compatible with the symplectic form $d\alpha$. Then one can define a boundary operator or *differential* ∂ on Θ, which satisfies the Leibnitz rule and reduces the degree by 1. We do not need the exact definition here. It suffices to say that ∂ is defined by counting, in an appropriate sense, the number of certain J-holomorphic curves in the symplectization $M \times \mathbf{R}$ of M. The properties of the differential graded algebra (Θ, ∂) are summarized in the following proposition (see [5]).

Theorem 3.1. (1) $\partial^2 = 0$.

(2) The graded *contact homology algebra* $H\Theta = \ker \partial / \operatorname{im} \partial$ is independent of the choices of J and α and is an invariant of the contact structure ξ up to isomorphism. □

Generally, contact homology is not easy to compute unless the differential ∂ vanishes, in which case $H\Theta = \Theta$. This happens, for instance, when all the generators of Θ are of even degree; that is, every periodic Reeb trajectory has an even reduced Conley-Zehnder index. In such a case, the number of periodic trajectories of each index gives an invariant of the contact structure. We use this remark in the next section to prove the main theorem.

4 Finding a generic perturbation

In this section, we prove Theorem 2.1. For this purpose, we explicitly construct a perturbation of the contact form α_p, which makes the Reeb flow nondegenerate. This allows us to compute the contact homology of the manifold (S^{4m+1}, ξ_p). It should be noted that the computations in this section could be considerably simplified by applying a "Morse-Bott"-type theory in the context of contact homology, which is yet to be developed.

Fix a natural number p such that $p \equiv \pm 1 \mod 8$ and denote

$$M = \Sigma\left(p, 2, \ldots, 2\right) \cong S^{4m+1}, \quad \alpha = \alpha_p, \quad \xi = \ker \alpha.$$

It is convenient to make the following unitary change of coordinates:

$$w_0 = z_0, \quad w_1 = z_1,$$
$$\begin{pmatrix} w_{2j} \\ w_{2j+1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} z_{2j} \\ z_{2j+1} \end{pmatrix}, \quad j \geq 1.$$

In these coordinates,

$$M = \left\{ w \in \mathbf{C}^{n+1} \mid w_0^p + w_1^2 + 2\sum_{j=1}^{m} w_{2j} w_{2j+1} = 0,\ |w|^2 = 1 \right\}.$$

Consider the function

$$H(w) = |w|^2 + \sum_{j=1}^{m} \varepsilon_j \left(|w_{2j}|^2 - |w_{2j+1}|^2\right), \quad \text{where } 0 < \varepsilon_j < 1.$$

Clearly, H is a real positive function on M, so $\alpha' = H^{-1}\alpha$ is a contact form on M, which defines the same contact structure ξ as α.

Lemma 4.1. The Reeb vector field of α' is

$$R_{\alpha'}(w) = \left(\frac{4i}{p} w_0, 2iw_1, 2i(1+\varepsilon_1)w_2, 2i(1-\varepsilon_1)w_3, \ldots, 2i(1+\varepsilon_m)w_{n-1}, 2i(1-\varepsilon_m)w_n\right). \quad \square$$

Proof. Denote

$$\omega = d\alpha = \frac{ip}{4} dw_0 \wedge d\overline{w}_0 + \frac{i}{2} \sum_{j=1}^{n} dw_j \wedge d\overline{w}_j.$$

This is a symplectic form on $\mathbf{C}^{n+1}$ with a Liouville vector field $X(w) = w/2$; i.e., $\alpha = i_X \omega$. Let us check that $R_{\alpha'}$ is equal to X_H, the Hamiltonian vector field of H with respect to ω. This vector field is defined by the equation $i_{X_H}\omega = -dH$ and is easily computed to be the one in the statement of the lemma.

It is straightforward to check that X_H is tangent to M. Now,

$$\alpha(X_H) = \omega(X, X_H) = -\omega(X_H, X) = dH(X) = H,$$

where the last equality follows from the fact that H is a quadratic Hamiltonian. Therefore,

$$\begin{aligned} i_{X_H} d\alpha' &= i_{X_H} d(H^{-1}\alpha) \\ &= i_{X_H}(H^{-1}\omega - H^{-2} dH \wedge \alpha) \\ &= -H^{-1} dH - H^{-2} \underbrace{dH(X_H)}_{0} \alpha + H^{-2}\alpha(X_H) dH \\ &= -H^{-1} dH + H^{-1} dH = 0, \\ i_{X_H}\alpha' &= H^{-1}\alpha(X_H) = 1, \end{aligned}$$

which proves our statement. ∎

Therefore, the flow generated by $R_{\alpha'}$ is

$$\varphi^t(w) = \left(e^{4it/p} w_0, e^{2it} w_1, e^{2it(1+\varepsilon_1)} w_2, e^{2it(1-\varepsilon_1)} w_3, \dots, e^{2it(1+\varepsilon_m)} w_{n-1}, e^{2it(1-\varepsilon_m)} w_n\right).$$

In particular, when the numbers ε_j are irrational and linearly independent over $\mathbf{Q}$, the only periodic trajectories of $R_{\alpha'}$ are

$$\gamma_0(t) = \left(re^{4it/p}, ir^{p/2}e^{2it}, 0, \dots, 0\right), \quad r > 0,\ r^p + r^2 = 1, \quad 0 \le t \le p\pi,$$

$$\gamma_j^+(t) = \left(0, \dots, 0, \underbrace{e^{2it(1+\varepsilon_j)}}_{2j}, 0, \dots, 0\right), \quad 0 \le t \le \frac{\pi}{1+\varepsilon_j}, \quad j = 1, \dots, m,$$

$$\gamma_j^-(t) = \left(0, \dots, 0, \underbrace{e^{2it(1-\varepsilon_j)}}_{2j+1}, 0, \dots, 0\right), \quad 0 \le t \le \frac{\pi}{1-\varepsilon_j}, \quad j = 1, \dots, m.$$

Of course, we must also consider all the multiples $N\gamma_0$, $N\gamma_j^\pm$ of these trajectories, where N is a natural number.

Lemma 4.2. Assume that ε_j are irrational and linearly independent over $\mathbf{Q}$. Then all periodic trajectories of $R_{\alpha'}$ are nondegenerate. Their reduced Conley-Zehnder indices are given by

$$\begin{aligned}\tilde{\mu}(N\gamma_0) &= 2Np(n-2) + 4N + n - 3,\\ \tilde{\mu}(N\gamma_j^{\pm}) &= 2\left\lfloor \frac{2N}{p(1\pm\varepsilon_j)} \right\rfloor + 2\left\lfloor \frac{N}{1\pm\varepsilon_j} \right\rfloor\\ &\quad + 2\sum_{\substack{k=1\\k\neq j}}^{m}\left(\left\lfloor \frac{N(1+\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor + \left\lfloor \frac{N(1-\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor\right) + 2n - 4.\end{aligned} \qquad \square$$

Proof. Under the usual identification of $T\mathbf{C}^{n+1}$ with $\mathbf{C}^{n+1}$, the contact plane ξ at a point $w \in M$ is given by

$$\xi_w = T_wM \cap \ker\alpha_w = \Bigg\{ v \in \mathbf{C}^{n+1} \mid pw_0^{p-1}v_0 + 2w_1v_1 + 2\sum_{j=1}^{m}(w_{2j}v_{2j+1} + w_{2j+1}v_{2j}) = 0,\ \langle w, v\rangle = 0,\ \alpha_w(v) = 0 \Bigg\}.$$

In particular,

$$\xi_{\gamma_0(0)} = \left\{ v \in \mathbf{C}^{n+1} \mid v_0 = v_1 = 0 \right\}.$$

The Poincaré return map of φ along γ_0 is a linear transformation of this space given by the matrix

$$\operatorname{diag}\left(e^{2\pi i p\varepsilon_1}, e^{-2\pi i p\varepsilon_1}, \ldots, e^{2\pi i p\varepsilon_m}, e^{-2\pi i p\varepsilon_m}\right).$$

Since the numbers $\varepsilon_1, \ldots, \varepsilon_m$ are irrational, this map and all its iterates have no eigenvalues equal to 1. Therefore, γ_0 and its multiples are nondegenerate trajectories. Similarly,

$$\xi_{\gamma_j^{\pm}(0)} = \left\{ v \in \mathbf{C}^{n+1} \mid v_{2j} = v_{2j+1} = 0 \right\},$$

and one can check in the same way that trajectories $N\gamma_j^{\pm}$ are nondegenerate.

For the computation of indices, we can equip ξ with the symplectic form $\omega = d\alpha$ instead of $d\alpha'$. This follows from the fact that $\omega|_{\xi} = Hd\alpha'|_{\xi}$ and H is constant along the Reeb trajectories of α'. Thus we consider ξ as a symplectic subbundle of $(T_M\mathbf{C}^{n+1}, \omega)$. At a point $w \in M$, the symplectic complement $\xi_w^{\perp}$ of ξ_w in $T_w\mathbf{C}^{n+1}$ is a 4-dimensional real vector space spanned by the vectors

$$\begin{aligned} X_1 &= (\overline{w}_0^{p-1}, \overline{w}_1, \overline{w}_3, \overline{w}_2, \ldots, \overline{w}_n, \overline{w}_{n-1}), \quad & Y_1 &= iX_1,\\ X_2 &= -i(2w_0/p, w_1, w_2, \ldots, w_{n-1}, w_n), \quad & Y_2 &= w.\end{aligned}$$

A straightforward computation shows

$$\begin{aligned}
\omega(X_1, X_2) = \omega(Y_1, X_2) &= 0,\\
\omega(X_1, Y_2) = \frac{p-2}{2}\mathrm{Im}(w_0^p), &\quad \omega(Y_1, Y_2) = -\frac{p-2}{2}\mathrm{Re}(w_0^p),\\
\omega(X_1, Y_1) = \frac{p}{2}|w_0|^{2p-2} + |w_1|^2 + \cdots + |w_n|^2 > 0, &\quad \omega(X_2, Y_2) = 1.
\end{aligned}$$

Now define

$$\begin{aligned}
\tilde{X}_1 &= \frac{X_1}{\sqrt{\omega(X_1, Y_1)}}, \quad \tilde{Y}_1 = \frac{Y_1}{\sqrt{\omega(X_1, Y_1)}} = i\tilde{X}_1,\\
\tilde{X}_2 &= X_2,\\
\tilde{Y}_2 &= Y_2 - \frac{\omega(X_1, Y_2)Y_1 - \omega(Y_1, Y_2)X_1}{\omega(X_1, Y_1)} \quad = \quad Y_2 - \frac{p-2}{2}\frac{w_0^p}{\omega(X_1, Y_1)}X_1.
\end{aligned}$$

This is a standard basis for the symplectic vector space $\xi_w^\perp$; i.e., the form ω in it is given by the matrix

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}.$$

In particular, the bundle $\xi^\perp$ is symplectically trivial. Extending the linearized Reeb flow φ_*^t to $T_M\mathbf{C}^{n+1}$ in the obvious way, we obtain

$$\begin{aligned}
\varphi_*^t\tilde{X}_1(w) &= e^{4it}\tilde{X}_1(\varphi^t w), \quad \varphi_*^t\tilde{Y}_1(w) = e^{4it}\tilde{Y}_1(\varphi^t w),\\
\varphi_*^t\tilde{X}_2(w) &= \tilde{X}_2(\varphi^t w), \quad \varphi_*^t\tilde{Y}_2(w) = \tilde{Y}_2(\varphi^t w).
\end{aligned}$$

A trivialization of ξ over any disc in M followed by the above trivialization of $\xi^\perp$ gives a trivialization of $T\mathbf{C}^{n+1}$, which is homotopic to the standard one. Thus, given a periodic trajectory γ of φ, the flow and its restrictions to ξ and $\xi^\perp$ define linear symplectic paths $\Phi^t \in \mathbf{Sp}(2n+2)$, $\Phi_1^t \in \mathbf{Sp}(2n-2)$, $\Phi_2^t \in \mathbf{Sp}(4)$ so that $\Phi^t = \Psi^t(\Phi_1^t \oplus \Phi_2^t)(\Psi^0)^{-1}$ for some contractible loop $\Psi^t \in \mathbf{Sp}(2n+2)$. Explicitly,

$$\begin{aligned}
\Phi^t &= \mathrm{diag}\left(e^{4it/p}, e^{2it}, e^{2it(1+\varepsilon_1)}, e^{2it(1-\varepsilon_1)}, \ldots, e^{2it(1+\varepsilon_m)}, e^{2it(1-\varepsilon_m)}\right),\\
\Phi_2^t &= \mathrm{diag}(e^{4it}, 1),
\end{aligned}$$

where $0 \le t \le Np\pi$ for $\gamma = N\gamma_0$ and $0 \le t \le N\pi/(1 \pm \varepsilon_j)$ for $\gamma = N\gamma_j^\pm$.

Recall that Robbin and Salamon in [11] defined the Conley-Zehnder index $\mu(\Phi)$ for arbitrary symplectic paths Φ, not just those ending in a matrix with eigenvalues different from 1. This index satisfies the following properties.

(i) $\mu(\Phi)$ is invariant under any homotopy of Φ with fixed end-points.

(ii) For any symplectic path Ψ, $\mu(\Psi\Phi\Psi^{-1}) = \mu(\Phi)$.

(iii) Identify $\mathbf{Sp}(2n_1) \oplus \mathbf{Sp}(2n_2)$ as a subgroup of $\mathbf{Sp}(2n_1 + 2n_2)$ in the obvious way. Then $\mu(\Phi_1 \oplus \Phi_2) = \mu(\Phi_1) + \mu(\Phi_2)$.

(iv) Let $\Phi(t) = e^{it} \in \mathbf{Sp}(2),\ 0 \le t \le T$. Then

$$\mu(\Phi) = \begin{cases} T/\pi, & T \in 2\pi\mathbf{Z}, \\ 2\left\lfloor T/2\pi \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

By (i), (ii), and (iii), the index of γ is given by

$$\mu(\gamma) = \mu(\Phi_1) = \mu(\Phi) - \mu(\Phi_2).$$

Using (iii) and (iv), we get

$$\begin{aligned}
\mu(N\gamma_0) &= 4N + 2Np + 2\sum_{k=1}^{m}(\lfloor Np(1+\varepsilon_k)\rfloor + \lfloor Np(1-\varepsilon_k)\rfloor + 1) - 4Np \\
&= 4N + 2Np + 2Np(n-1) - 4Np = 2Np(n-2) + 4N, \\
\mu(N\gamma_j^{\pm}) &= 2\left\lfloor \frac{2N}{p(1\pm\varepsilon_j)} \right\rfloor + 2\left\lfloor \frac{N}{1\pm\varepsilon_j} \right\rfloor + 2\left\lfloor \frac{N(1\mp\varepsilon_j)}{1\pm\varepsilon_j} \right\rfloor + 3 + 2N \\
&\quad + 2\sum_{k\neq j}\left(\left\lfloor \frac{N(1+\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor + \left\lfloor \frac{N(1-\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor + 1\right) - 2\left\lfloor \frac{2N}{1\pm\varepsilon_j} \right\rfloor - 1 \\
&= 2\left\lfloor \frac{2N}{p(1\pm\varepsilon_j)} \right\rfloor + 2\left\lfloor \frac{N}{1\pm\varepsilon_j} \right\rfloor \\
&\quad + 2\sum_{k\neq j}\left(\left\lfloor \frac{N(1+\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor + \left\lfloor \frac{N(1-\varepsilon_k)}{1\pm\varepsilon_j} \right\rfloor\right) + n - 1.
\end{aligned}$$

Since, by definition, $\tilde{\mu} = \mu + n - 3$, the claim now follows. ■

From the statement of the lemma, we get that all the (reduced) indices are even numbers starting with $2n - 4$. Therefore, by the argument at the end of the previous section, contact homology is the polynomial algebra generated by periodic Reeb trajectories. In particular, for any integer k, the number c_k of trajectories of index k does not depend on the choice of parameters ε_j (it would be interesting to give a purely number-theoretical proof of this fact).

Lemma 4.3. The numbers c_k are given by

$$c_k = \begin{cases} 0, & k \text{ is odd or } k < 2n-4, \\ 2, & k = 2\lfloor 2N/p \rfloor + 2(N+1)(n-2),\ N \geq 1,\ 2N+1 \notin p\mathbf{Z}, \\ 1, & \text{in all other cases.} \end{cases}$$

□

Proof. Assume $0 < \varepsilon_1 < \varepsilon_2 < \cdots \varepsilon_m < 1$ and denote

$$\tilde{\mu}^{\pm}_{N,j} = \lim_{\varepsilon_m \to 0} \tilde{\mu}(N\gamma^{\pm}_j), \qquad \tilde{\mu}_{N,0} = \tilde{\mu}(N\gamma_0).$$

A rather crude estimate gives $\tilde{\mu}(N\gamma^{\pm}_j) > N$; thus, to compute c_k, we need to consider only finitely many numbers $\tilde{\mu}(N\gamma^{\pm}_j)$ with $N < k$. Taking ε_m small enough, we get $\tilde{\mu}(N\gamma^{\pm}_j) = \tilde{\mu}^{\pm}_{N,j}$. Since for any $x \in \mathbf{R}$,

$$\lim_{\varepsilon \to 0+} \lfloor x + \varepsilon \rfloor = \lfloor x \rfloor, \qquad \lim_{\varepsilon \to 0+} \lfloor x - \varepsilon \rfloor = \lceil x - 1 \rceil,$$

we obtain

$$\begin{aligned} \tilde{\mu}^{+}_{N,j} &= 2\left\lceil \frac{2N}{p} - 1 \right\rceil + 2(N-1) + 2\big((j-1)(N-1) + (m-j)N \\ &\quad + (m-1)(N-1)\big) + 2n - 4 = 2\left\lceil \frac{2N}{p} - 1 \right\rceil + 2N(n-2) + n - 1 - 2j, \\ \tilde{\mu}^{-}_{N,j} &= 2\left\lfloor \frac{2N}{p} \right\rfloor + 2N + 2\big((m-1)N + (j-1)N + (m-j)(N-1)\big) \\ &\quad + 2n - 4 = 2\left\lfloor \frac{2N}{p} \right\rfloor + 2N(n-2) + n - 3 + 2j. \end{aligned}$$

For any N,

$$\tilde{\mu}^{+}_{N,m} < \tilde{\mu}^{+}_{N,m-1} < \cdots < \tilde{\mu}^{+}_{N,1} < \tilde{\mu}^{-}_{N,1} < \cdots < \tilde{\mu}^{-}_{N,m-1} < \tilde{\mu}^{-}_{N,m},$$

and the numbers in this sequence increase by differences of 2 with only one exception: when $p|N$, one has

$$\tilde{\mu}^{+}_{N,1} + 2 = \tilde{\mu}_{N/p,0} = \tilde{\mu}^{-}_{N,1} - 2.$$

In addition,

$$\tilde{\mu}^{+}_{N+1,m} - \tilde{\mu}^{-}_{N,m} = \begin{cases} 0, & \lfloor 2N/p \rfloor = \lceil 2(N+1)/p - 1 \rceil, \\ 2, & \lfloor 2N/p \rfloor < \lceil 2(N+1)/p - 1 \rceil. \end{cases}$$

Therefore, c_k is equal to 1 or 2 for all even $k \geq 2n-4$, and 0 otherwise. The only case in which $c_k = 2$ is when

$$k = \tilde{\mu}^{-}_{N,m} = 2\left\lfloor \frac{2N}{p} \right\rfloor + 2(N+1)(n-2) \quad \text{and} \quad \left\lfloor \frac{2N}{p} \right\rfloor = \left\lceil \frac{2(N+1)}{p} - 1 \right\rceil.$$

The second condition is equivalent to $2N + 1 \notin p\mathbf{Z}$. ■

Let us list the first few terms of the sequence c_k:

$$\overset{2(n-2)}{1}, 0, 1, 0, \ldots, 1, 0, \overset{4(n-2)}{2}, 0, 1, 0, \ldots, 1, 0, \overset{6(n-2)}{2}, 0, 1, \ldots, \overset{(p+1)(n-2)}{1}, 0, 1, 0 \ldots.$$

We see that the first irregularity occurs at index $(p+1)(n-2)$: instead of two periodic trajectories of that index, there is only one. Thus for different values of p, the contact structures ξ_p have different contact homology invariants, and so are not isomorphic. This proves Theorem 2.1.

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